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LETTER TO THE EDITOR

Generating function for the product of the associated Laguerre and Hermite polynomials

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Abstract. The generating function for the product of the associated Laguerre and Hermite polynomials is formulated and used to determine the coordinate-space wavefunction of the displaced number states of the harmonic oscillator for the general case of a complex displacement parameter. The benefits, in this context, of an extended form of the associated Laguerre polynomials are advertised.

In this letter we draw attention to a useful relation satisfied by the associated Laguerre and Hermite polynomials which finds its application in determination of the coordinate-space wavefunction of the *displaced number states* of the harmonic oscillator [1–4]. The importance and unusual properties of these states have been discussed recently, in some detail, in [5–6]. It is hoped that these states can be prepared, in the realm of quantum optics, by driving the microwave cavity field (initially prepared in a number state) of the micromaser, by a classical current [5].

The relation of interest, mentioned in the opening paragraph, is the following generating function for the product of the associated Laguerre and Hermite polynomials

$$\frac{s^m}{m!} H_m(\xi - 2\,\mathrm{Re}\{s\}) \exp(-s^2 + 2s\xi) = \sum_{n=0}^{\infty} \frac{L_m^{(n-m)}(2|s|^2) H_n(\xi)}{n!} s^n. \tag{1}$$

Here, s is complex, ξ real, $m = 0, 1, 2, \ldots$ while H_n and $L_m^{(k)}$ denote the Hermite and associated Laguerre polynomials respectively. In the latter case one has explicitly

$$L_m^{(k)}(x) = \sum_{l=\max(0,-k)}^{m} \binom{m+k}{m-l} \frac{(-x)^l}{l!}. \tag{2}$$

When $k \geqslant 0$, equation (2) *agrees with the usual expression for the associated Laguerre* polynomials given, for example, in [7]. Additionally, equation (2) is valid for *negative* k ($\geqslant -m, m \geqslant 0$), thus representing a slight extension of the usual expression. One has, for example

$$L_2^{(0)}(x) = \tfrac{1}{2}x^2 - 2x + 1 \qquad L_2^{(1)}(x) = \tfrac{1}{2}x^2 - 3x + 3$$

$$L_2^{(2)}(x) = \tfrac{1}{2}x^2 - 4x + 6 \qquad L_2^{(3)}(x) = \tfrac{1}{2}x^2 - 5x + 10$$

as usual; additionally

$$L_2^{(-1)}(x) = \tfrac{1}{2}x^2 - x \qquad L_2^{(-2)}(x) = \tfrac{1}{2}x^2.$$

By adopting equation (2) as a *definition* of the associated Laguerre polynomials one achieves, as we shall see, in the case of displaced number states, unified and simplified treatment. Equation (2) is, in fact, equivalent to the expression used by Perelomov in his book [8]. Formula (2) for negative k is, of course, already known [9].

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Firstly, we note that in the special case $m=0$, equation (1) reduces to the well known generating function for the Hermite polynomials (see, for example, [10]). Relation (1) can be proved by induction. The proof is based on the following two recurrence relations

$$L_m^{(k-1)}(x)-L_m^{(k)}(x)+L_{m-1}^{(k)}(x)=0 \tag{3}$$

$$(m+1)L_{m+1}^{(k-1)}(x)-(m+k)L_m^{(k-1)}(x)+xL_m^{(k)}(x)=0 \tag{4}$$

which, in turn, can be proved directly with the help of (2).

Secondly, we use the generating function, equation (1), to determine the coordinate-space wavefunction of the displaced number state of the harmonic oscillator for the general case of a *complex* displacement parameter. As is well known, the general solution of the relevant time-dependent Schrödinger equation reads

$$\Psi(x,t)=\sum_{n=0}^{\infty} c_n \exp\left(-\frac{\mathrm{i}}{\hbar}E_n t\right)u_n(x). \tag{5}$$

Here $u_n(x)$ denotes the harmonic-oscillator eigenfunction corresponding to the nth energy level E_n [10]. When we take [1, 4-6]

$$c_n \to c_n^{(m)} \equiv \left(\frac{m!}{n!}\right)^{1/2} \alpha^{n-m} \exp\left(-\frac{|\alpha|^2}{2}\right) L_m^{(n-m)}(|\alpha|^2) \tag{6}$$

we obtain the displaced number state $\Psi_\alpha^{(m)}(x,t)$. Note that, by adopting (2) as a definition of the associated Laguerre polynomial, we need only one expression for $c_n^{(m)}$ which is valid for any $n-m$ (positive or negative). This should be compared with [6] (for example) where *two* different expressions for $c_n^{(m)}$ had to be used. The relation (1), then, also gains in simplicity. In (6), $\alpha\equiv|\alpha|\,\mathrm{e}^{\mathrm{i}\varphi}$ is the complex displacement parameter. By inserting (6) into (5), using the well known expression for $u_n(x)$ together with (1), and with $\xi\equiv x/(2^{1/2}\sigma)$, $s\equiv\alpha\,\mathrm{e}^{-\mathrm{i}\omega t}/2^{1/2}$, we obtain the following general expression for the coordinate space wavefunction representing displaced number state

$$\Psi_\alpha^{(m)}(x,t)=\mathrm{e}^{\mathrm{i}\phi_m}u_m(x-x_c(t)). \tag{7}$$

Here, the phase $\phi_m=\phi_m(x,t)$, is given by

$$\phi_m(x,t)=-(m+\tfrac{1}{2})\omega t+\frac{|\alpha|^2}{2}\sin 2(\omega t-\varphi)-\frac{|\alpha|x}{\sigma}\sin(\omega t-\varphi) \tag{8}$$

with

$$x_c(t)=\langle x\rangle=2\sigma|\alpha|\cos(\omega t-\varphi) \tag{9}$$

and

$$\sigma\equiv(\hbar/2m\omega)^{1/2} \tag{10}$$

denoting the half-width of the harmonic-oscillator ground state $u_0(x)$. The form of (7) is very perspicuous; $\Psi_\alpha^{(m)}(x,t)$ is, to within a phase factor, equal to the harmonic-oscillator eigenfunction u_m displaced to the point $x_c(t)$ which follows the classical motion of the harmonic oscillator with the amplitude $2\sigma|\alpha|$. The shape of the wavepacket remains the same all the time. Equation (7) reduces to the wavefunction given previously in [5] for the special case $t=0$ and $\varphi=0$ (i.e. for *real* displacement parameter $\alpha=|\alpha|$). For $m=0$, $\Psi_\alpha^{(m)}(x,t)$ reduces to the familiar *coherent* state wavefunction. Finally, let us remark that the displaced number states have been related recently [3–4] to the general form of the transition probability for quantum oscillator driven by external force (this was found independently some time ago by Feynman [11, 12] and Schwinger [13]). As is well known, this probability is given by $|c_n^{(m)}|^2$. We see that single quantum oscillator driven by a suitable transient external force evolves from the initial state $u_m(x)$ to the final, displaced number state with the wavefunction given (to within a phase factor) by $\Psi_\alpha^{(m)}(x, t)$, equation (7). In particular, by perturbing the quantum oscillator initially in the ground state, by a transient classical force, one obtains the corresponding coherent state [14].

In conclusion, in this letter we have formulated the generating function for the product of the associated Laguerre and Hermite polynomials and discussed its application to displaced number states of the harmonic oscillator. We commented briefly on the benefits of a slightly generalized form of the associated Laguerre polynomials in this context. In particular, we showed how it becomes possible, with the help of the generating function (1), to extract the time dependent wavefunction for displaced number state in closed form for the general case of a *complex* displacement parameter.

References

[1] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
[2] Boiteux M and Levelut A 1973 *J. Phys. A: Math. Gen.* **6** 589
[3] Roy S M and Singh V 1982 *Phys. Rev.* D **25** 3413
[4] Venkata Satyanarayana M 1985 *Phys. Rev.* D **32** 400
[5] de Oliveira F A M, Kim M S, Knight P L and Bužek V 1990 *Phys. Rev.* A **41** 2645
[6] Brisudova M 1991 *J. Mod. Opt.* **38** 2505
[7] Gradshteyn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York: Academic)
[8] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[9] Erdelyi A *et al* 1953 *Higher Transcendental Functions* (New York: McGraw-Hill) vol 2, p 189, equation (14)
[10] Schiff L I 1949 *Quantum Mechanics* (New York: McGraw-Hill)
[11] Feynman R P 1950 *Phys. Rev.* **80** 440
[12] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[13] Schwinger J 1953 *Phys. Rev.* **91** 728
[14] Carruthers P and Nieto M M 1965 *Am. J. Phys.* **33** 537